# Impossibility of partial swapping of Quantum Information

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**Abstract**

It is a well known fact that a quantum state $|\psi(\theta, \phi)\rangle$ is represented by a point on the Bloch sphere, characterized by two parameters $\theta$ and $\phi$. Here in this work, we find out another impossible operation in quantum information theory . We name this impossibility as 'Impossibility of partial swapping of quantum information '. By this we mean that if two unknown quantum states are given at the input port, there exists no physical process , consistent with the principles of quantum mechanics, by which we can partially swap either of the two parameters $\theta$ and $\phi$ between these two quantum states. In this work we provided the impossibility proofs for the qubits( i.e the quantum states taken from two dimensional Hilbert space) and this impossible operation can be shown to hold in higher dimension also.

## 1  Introduction:

In quantum information theory it is most important to know various differences between the classical and quantum information. Quantum information theory on one hand broadens the set consisting of information processing protocols, but also on other hand puts

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restrictions on various operations which are feasible in classical information theory [1-3]. These restrictions on many quantum operations is making quantum information more secure. In the famous land mark paper of Wootters and Zurek it was shown that a single quantum cannot be cloned [1]. Later it was also shown by Pati and Braunstein that we cannot delete either of the two quantum states when we are provided with two identical quantum states at our input port [2]. In spite of these two famous 'no-cloning' [1] and 'no-deletion' [2] theorem there are many other 'no-go' theorems like 'no-self replication' [3] , 'no-partial erasure' [4], 'no-splitting' [5] and many more which have come up.

In this work we claim another impossibility in quantum information theory. This 'no-go' theorem is yet another addition to the set of forbidden operations in quantum information theory. We know that the quantum state $|\psi(\theta, \phi)\rangle$ can be represented as a point on the Bloch-sphere, characterized by two parameters $\theta$ and $\phi$. Here in this work we define a concept called partial swapping of quantum information, which is not the same as the swapping of two quantum states. As the information content in a qubit is dependent on the angles $\theta$ and $\phi$, the partial swapping of quantum parameters $\theta$ and $\phi$ is given by,

$$|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow |A(\theta_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \phi_1)\rangle \qquad (1)$$

$$|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow |A(\bar{\theta}_1, \phi_1)\rangle|\bar{A}(\theta_1, \bar{\phi}_1)\rangle \qquad (2)$$

Here in this work we will show that the above transformations will not be possible in quantum regime. The linear structure of quantum mechanics prohibits the execution of these operations. In the next section we will provide the impossibility proof of partial swapping of quantum parameters $\{\theta, \phi\}$ for two dimensional quantum states (i.e qubits) and also discussed about what will be the probable extension of the proof for quantum states taken from a general 'd' dimensional Hilbert space. Then the conclusion follows.

## 2    Partial swapping of quantum information: Linearity of quantum mechanics

Let us a consider unknown qubits $\{|X(\theta, \phi)\rangle, |\bar{X}(\bar{\theta}, \bar{\phi})\rangle\} \in H$, which can be expressed as a linear combination of basis vectors $\{|A(\theta_1, \phi_1)\rangle, |\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\}$ of the two dimensional

Hilbert space $H$.

$$|X(\theta, \phi)\rangle = a|A(\theta_1, \phi_1)\rangle + b|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle$$
$$|\bar{X}(\bar{\theta}, \bar{\phi})\rangle = c|A(\theta_1, \phi_1)\rangle + d|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \tag{3}$$

where,

$$a^2 + |b|^2 = 1$$
$$c^2 + |d|^2 = 1 \tag{4}$$

(where $a^2, b^2$ and $c^2, d^2$ are corresponding probabilities of the qubits $|X(\theta, \phi)\rangle, |\bar{X}(\bar{\theta}, \bar{\phi})\rangle$ to collapse into the states $\{|A(\theta_1, \phi_1)\rangle, |\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\}$, when it undergoes a measurement in the same basis). First of all let us assume that swapping of phase angles $\{\phi_1, \bar{\phi}_1\}$ be possible for the orthogonal basis vectors $\{|A(\theta_1, \phi_1)\rangle, |\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\}$.

$$|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle \longrightarrow |A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle$$
$$|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow |A(\theta_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \phi_1)\rangle$$
$$|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|A(\theta_1, \phi_1)\rangle \longrightarrow |\bar{A}(\bar{\theta}_1, \phi_1)\rangle|A(\theta_1, \bar{\phi}_1)\rangle$$
$$|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow |\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \tag{5}$$

Then the action of the above transformations defined for the orthogonal basis vectors, on the combined system of qubits $|X(\theta, \phi)\rangle, |\bar{X}(\bar{\theta}, \bar{\phi})\rangle$, is given by

$$|X(\theta, \phi)\rangle|\bar{X}(\bar{\theta}, \bar{\phi})\rangle =$$
$$\{a|A(\theta_1, \phi_1)\rangle + b|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\}\{c|A(\theta_1, \phi_1)\rangle + d|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\} =$$
$$ac|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle + ad|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle +$$
$$bc|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|A(\theta_1, \phi_1)\rangle + bd|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow$$
$$ac|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle + ad|A(\theta_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \phi_1)\rangle +$$
$$bc|\bar{A}(\bar{\theta}_1, \phi_1)\rangle|A(\theta_1, \bar{\phi}_1)\rangle + bd|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \tag{6}$$

However if we consider the qubit $|X(\theta, \phi)\rangle$ and it's orthogonal state $|\bar{X}(\bar{\theta}, \bar{\phi})\rangle$ at the input

port, then the perfect swapping of phase angles between these two states will be given by the transformation,

$$|X(\theta,\phi)\rangle|\bar{X}(\bar{\theta},\bar{\phi})\rangle \longrightarrow |X(\theta,\bar{\phi})\rangle|\bar{X}(\bar{\theta},\phi)\rangle =$$
$$\{\alpha|A(\theta_1,\phi_1)\rangle + \beta|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\}\{\gamma|A(\theta_1,\phi_1)\rangle + \delta|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\} =$$
$$\alpha\gamma|A(\theta_1,\phi_1)\rangle|A(\theta_1,\phi_1)\rangle + \alpha\delta|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle +$$
$$\beta\gamma|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|A(\theta_1,\phi_1)\rangle + \beta\delta|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle \tag{7}$$

where

$$\alpha^2 + |\beta|^2 = 1$$
$$\gamma^2 + |\delta|^2 = 1 \tag{8}$$

It is clearly evident that the equations (6) and (7) are not identical for all values of the two parameters with full generality. This indicates that linear structure of quantum mechanics doesn't allow swapping of the phase angles for unknown orthogonal qubits.

Next we will see see that quite alike to the previous case one cannot swap the azimuthal angles between two unknown orthogonal qubits $|X(\theta,\phi)\rangle, |\bar{X}(\bar{\theta},\bar{\phi})\rangle$. Let us once again assume that the swapping of the azimuthal angles is possible for orthogonal vectors $\{|A(\theta_1,\phi_1)\rangle, |\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\}$,

$$|A(\theta_1,\phi_1)\rangle|A(\theta_1,\phi_1)\rangle \longrightarrow |A(\theta_1,\phi_1)\rangle|A(\theta_1,\phi_1)\rangle$$
$$|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle \longrightarrow |A(\bar{\theta}_1,\phi_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle$$
$$|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|A(\theta_1,\phi_1)\rangle \longrightarrow |\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|A(\bar{\theta}_1,\phi_1)\rangle$$
$$|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle \longrightarrow |\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle \tag{9}$$

Now if we apply this swapping transformation defined for the orthogonal basis vectors $\{|A(\theta_1,\phi_1)\rangle, |\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\}$, on the product state $|X(\theta,\phi)\rangle|\bar{X}(\bar{\theta},\bar{\phi})\rangle$, the output is given by,

$$|X(\theta,\phi)\rangle|\bar{X}(\bar{\theta},\bar{\phi})\rangle =$$
$$\{a|A(\theta_1,\phi_1)\rangle + b|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\}\{c|A(\theta_1,\phi_1)\rangle + d|\bar{A}(\bar{\theta}_1,\bar{\phi}_1)\rangle\} =$$

$$ac|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle + ad|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle +$$
$$bc|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|A(\theta_1, \phi_1)\rangle + bd|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \longrightarrow$$
$$ac|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle + ad|A(\bar{\theta}_1, \phi_1)\rangle|\bar{A}(\theta_1, \bar{\phi}_1)\rangle +$$
$$bc|\bar{A}(\theta_1, \bar{\phi}_1)\rangle|A(\bar{\theta}_1, \phi_1)\rangle + bd|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \tag{10}$$

Now if we consider the swapping of the azimuthal angle of two unknown orthogonal qubits $|X(\theta, \phi)\rangle, |\bar{X}(\bar{\theta}, \bar{\phi})\rangle$, the final state after the swapping is given by,

$$|X(\theta, \phi)\rangle|\bar{X}(\bar{\theta}, \bar{\phi})\rangle \longrightarrow |X(\bar{\theta}, \phi)\rangle|\bar{X}(\theta, \bar{\phi})\rangle =$$
$$\{\alpha_1|A(\theta_1, \phi_1)\rangle + \beta_1|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\}\{\gamma_1|A(\theta_1, \phi_1)\rangle + \delta_1|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle\} =$$
$$\alpha_1\gamma_1|A(\theta_1, \phi_1)\rangle|A(\theta_1, \phi_1)\rangle + \alpha_1\delta_1|A(\theta_1, \phi_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle +$$
$$\beta_1\gamma_1|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|A(\theta_1, \phi_1)\rangle + \beta_1\delta_1|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle|\bar{A}(\bar{\theta}_1, \bar{\phi}_1)\rangle \tag{11}$$

where

$$\alpha_1^2 + |\beta_1|^2 = 1$$
$$\gamma_1^2 + |\delta_1|^2 = 1 \tag{12}$$

Again we observe that equations (10) and (11) are not identical with full generality. It is clearly evident that we cannot swap the azimuthal angles of two unknown orthogonal qubits.

The above proof only deals with qubits, however for a more general structure one should consider qudits. Arbitrary quantum states $|\psi(\theta, \phi)\rangle, |\bar{\psi}(\bar{\theta}, \bar{\phi})\rangle \in H$, (where $H$ is the d dimensional Hilbert space) can be expressed in terms of the basis vectors $\{|A_i(\theta_i, \phi_i)\rangle, i = 1, ....d\}$. Even if we assume that we can partially swap the phase angles and the azimuthal angles for the basis vectors $\{|A_i(\theta_i, \phi_i)\rangle, |A_j(\theta_j, \phi_j)\rangle\}$, then on the basis of the principle of linearity of quantum mechanics it is not possible to partially swap the same parameters for arbitrary quantum states $|\psi(\theta, \phi)\rangle, |\bar{\psi}(\bar{\theta}, \bar{\phi})\rangle$. This is because of the fact that the states obtained after partial swapping of the parameters of the two states $|\psi(\theta, \phi)\rangle, |\bar{\psi}(\bar{\theta}, \bar{\phi})\rangle$ is not going to be identical with the output obtained if we consider the linear combination of the two states in terms of their basis vectors and apply partial swapping for the basis vectors.

# 3  Conclusion:

In summary we can say that here we have introduced a new kind of impossible operation in quantum information theory, which we refer as the partial swapping of quantum parameter. It has already been observed that many quantum information processing protocol which are Hilbert space or for the equatorial states of the Bloch sphere , are not true for the entire complex Hilbert space. These operations are like remote state preparation [6], rotation of real angles in Grover's algorithm [7] and many. This no go theorem tells us that one cannot separately swap two parameters $\theta$ and $\phi$. This theorem provides a strong evidence that all such tasks that are possible in real Hilbert space can never be implemented on the entire complex Hilbert space.

# 4  Acknowledgement:

# 5  Reference:

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